ON TWISTED ALEXANDER POLYNOMIALS ASSOCIATED TO $SL_2(\mathbb{C})$-REPRESENTATIONS OF KNOT GROUPS

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ABSTRACT. We study the problem of determining the fiberedness and genus of a knot by twisted Alexander polynomials from the viewpoint of $SL_2(\mathbb{C})$-representations of the knot group. For a non-fibered two-bridge knot, it was shown in [KmM] that there exists an irreducible component of the non-abelian character variety of the knot group where all but finitely many characters determine the fiberedness and genus of the knot. In this note, we calculate the number of conjugacy classes of non-abelian $SL_2(\mathbb{C})$-representations of the knot group whose associated twisted Alexander polynomials do not determine the fiberedness and genus of the knot for certain classes of non-fibered two-bridge knots.

1. MAIN RESULTS

The twisted Alexander polynomial was introduced by Lin [Li] for knots in the 3-sphere and by Wada [Wa] for finitely presentable groups. It is a generalization of the classical Alexander polynomial and gives a powerful tool in low dimensional topology. One of the most important applications is the determination of fiberedness [FV2] and genus (the Thurston norm) [FV3] of knots by the twisted Alexander polynomials corresponding to finite representations. For literature on other applications and related topics, we refer to the survey paper by Friedl and Vidussi [FV1].

Following [KmM], we study the problem of determining the fiberedness and genus $g(K)$ of a knot K by twisted Alexander polynomials from the viewpoint of $SL_2(\mathbb{C})$-representations of the knot group. For each representation of the knot group into $SL_2(\mathbb{C})$, one can associate a rational function called the twisted Alexander polynomial, see Section 2. A representation (and its character) is said to determine the fiberedness of K if the coefficient of the highest degree term of the associated twisted Alexander polynomial is one if and only if K is a fibered knot. Moreover, we say that a representation (and its character) determines the knot genus if the degree of the associated twisted Alexander polynomial equals $4g(K) - 2$. It is known that every non-abelian representation of a fibered knot is monic [GKM] and determines the knot genus [KtM]. However, it is still unknown whether the converse holds. More precisely, we do not know if every non-abelian representation of a knot is monic and determines the knot genus then the knot is fibered or not. For a non-fibered two-bridge knot, Kim and Morifuji showed in [KmM] that there exists an irreducible component of the non-abelian character variety of the knot group where all but finitely many characters determine the fiberedness and genus of the knot. In this note, we exactly calculate the number of conjugacy classes of non-abelian representations of the knot group whose associated twisted Alexander polynomials do not determine the fiberedness and genus of the knot for certain classes of non-fibered two-bridge knots.

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For a non-fibered knot K, denote by $f_{\text{nab}}(K)$ (resp. $g_{\text{nab}}(K)$) be the number of conjugacy classes of non-abelian $SL_2(\mathbb{C})$-representations of the knot group whose associated twisted Alexander polynomials do not determine the fiberedness (resp. genus) of K.

Let $\mathfrak{b}(p,m)$ be the two-bridge knot associated to a pair of relatively prime integers $p > m > 1$ (see e.g. [BZ]) and K_m, $m > 1$, the m-twist knot (see Figures 1 and 2). Then we have the following.

Theorem 1. *For $K = \mathfrak{b}(6n+1,3)$, one has $f_{\text{nab}}(K) = g_{\text{nab}}(K) = 2n$.*

Theorem 2. *(i) For $K = K_{2n}$, one has $g_{\text{nab}}(K) = 1+(-1)^n$ and $f_{\text{nab}}(K) = 2n-2-a_n-b_n$ where*

$$a_n = \begin{cases} 2, & n \equiv 1 \pmod 6 \\ 0, & \text{otherwise} \end{cases},$$

$$b_n = \begin{cases} 2, & n \equiv 1 \pmod 5 \\ 0, & \text{otherwise} \end{cases}.$$

(ii) For $K = K_{2n-1}$, one has $g_{\text{nab}}(K) = 1+(-1)^n$ and $f_{\text{nab}}(K) = 2n-2-c_n-d_n-e_n$ where

$$c_n = \begin{cases} 1, & n \equiv 1 \pmod 3 \\ 0, & \text{otherwise} \end{cases},$$

$$d_n = \begin{cases} 2, & n \equiv 5 \pmod 6 \\ 0, & \text{otherwise} \end{cases},$$

$$e_n = \begin{cases} 2, & n \equiv 4 \pmod 5 \\ 0, & \text{otherwise} \end{cases}.$$

The note is organized as follows. We review the definition of twisted Alexander polynomials and some related work on fibering and genus of knots in Section 2. We then prove Theorem 1 in Section 2 and Theorem 2 in Section 3.

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2. Twisted Alexander polynomials

Let K be a knot and $G_K = \pi_1(S^3 \backslash K)$ its knot group. We choose and fix a Wirtinger presentation

$$G_K = \langle a_1, \dots, a_\ell \mid r_1, \dots, r_{\ell-1} \rangle.$$

Then the abelianization homomorphism $f : G_K \to H_1(S^3\backslash K, \mathbb{Z}) \cong \mathbb{Z} = \langle t \rangle$ is given by $f(a_1) = \cdots = f(a_\ell) = t$. Here we specify a generator t of $H_1(S^3\backslash K; \mathbb{Z})$ and denote the sum in $\mathbb{Z}$ multiplicatively. Let us consider a linear representation $\rho : G_K \to SL_2(\mathbb{C})$.

These maps naturally induce two ring homomorphisms $\widetilde{\rho} : \mathbb{Z}[G_K] \to M(2,\mathbb{C})$ and $\widetilde{f} : \mathbb{Z}[G_K] \to \mathbb{Z}[t^{\pm 1}]$, where $\mathbb{Z}[G_K]$ is the group ring of G_K and $M(2,\mathbb{C})$ is the matrix algebra of degree 2 over $\mathbb{C}$. Then $\widetilde{\rho} \otimes \widetilde{f}$ defines a ring homomorphism $\mathbb{Z}[G_K] \to M\left(2, \mathbb{C}[t^{\pm 1}]\right)$. Let F_ℓ denote the free group on generators $a_1, \dots, a_\ell$ and $\Phi : \mathbb{Z}[F_\ell] \to M\left(2, \mathbb{C}[t^{\pm 1}]\right)$ the

composition of the surjection $\mathbb{Z}[F_q] \to \mathbb{Z}[G_K]$ induced by the presentation of G_K and the map $\widetilde{\rho} \otimes \widetilde{f} : \mathbb{Z}[G_K] \to M(2, \mathbb{C}[t^{\pm 1}])$.

Let us consider the $(\ell-1) \times \ell$ matrix M whose (i,j)-component is the 2×2 matrix

$$\Phi\left(\frac{\partial r_i}{\partial a_j}\right) \in M\left(2, \mathbb{Z}[t^{\pm 1}]\right),$$

where $\partial / \partial a$ denotes the free differential calculus. For $1 \leq j \leq \ell$, let us denote by M_j the $(\ell-1) \times (\ell-1)$ matrix obtained from M by removing the jth column. We regard M_j as a $2(\ell-1) \times 2(\ell-1)$ matrix with coefficients in $\mathbb{C}[t^{\pm 1}]$. Then Wada's twisted Alexander polynomial of a knot K associated to a representation $\rho : G_K \to SL_2(\mathbb{C})$ is defined to be a rational function

$$\Delta_{K,\rho}(t) = \frac{\det M_j}{\det \Phi(1 - a_j)}$$

and moreover well-defined up to a factor t^{2k} ($k \in \mathbb{Z}$), see [Wa].

A representation $\rho : G_K \to SL_2(\mathbb{C})$ is called non-abelian if $\rho(G_K)$ is a non-abelian subgroup of $SL_2(\mathbb{C})$. Suppose ρ is a non-abelian representation. It is known that the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ associated to ρ is always a Laurent polynomial for any knot K [KtM], and is a monic polynomial if K is a fibered knot. It is also known that the converse holds for alternating knots [KmM]. Moreover, if K is a knot of genus g then $\deg(\Delta_{K,\rho}(t)) \leq 4g - 2$ [FK], and the equality holds if K is fibered [KtM].

We say that the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ determines the fiberedness of K if it is a monic polynomial if and only if K is fibered. We also say $\Delta_{K,\rho}(t)$ determines the knot genus $g(K)$ if $\deg(\Delta_{K,\rho}(t)) = 4g(K) - 2$ holds.

3. Proof of Theorem 1

The following result is well-known.

Lemma 3.1. $\mathfrak{b}(p,3)$ *is non-fibered if and only if* $p \equiv 1 \pmod 6$.

Proof. The proof is based on the calculation of the Alexander polynomial. Since $\mathfrak{b}(p,3)$ is an alternating knot, it is fibered if and only if its Alexander polynomial is monic, i.e. has leading coefficient 1, see [Cr, Mu]. The leading coefficient of the Alexander polynomial of $\mathfrak{b}(p,3)$ is 2 if $p \equiv 1 \pmod 6$ and is 1 if $p \equiv -1 \pmod 6$, see e.g. [HS, Section 2]. The lemma follows. $\square$

The standard presentation of the knot group of $K = \mathfrak{b}(p,m)$ is $G_K = \langle a, b \mid wa = bw \rangle$ where $w = a^{\varepsilon_1} b^{\varepsilon_2} \dots a^{\varepsilon_{p-2}} b^{\varepsilon_{p-1}}$ and $\varepsilon_j = (-1)^{\lfloor jq/p \rfloor}$, see e.g. [BZ]. Here a and b are 2 standard generators of a two-bridge knot group.

For a representation $\rho : G_K \to SL_2(\mathbb{C})$, let $x = \operatorname{tr} \rho(a) = \operatorname{tr} \rho(b)$ and $z = \operatorname{tr} \rho(ab)$. Let $d = \frac{p-1}{2}$. By [Le], the representation ρ is non-abelian if and only if $R_w(x,z) = 0$ where

$$R_w(x,z) = \operatorname{tr} \rho(w) - \operatorname{tr} \rho(w') + \cdots + (-1)^{d-1} \operatorname{tr} \rho(w^{(d-1)}) + (-1)^d.$$

Here if u is a word in the letters a, b then u' denotes the word obtained from u by deleting the two letters at the two ends. Then $w^{(d-1)}$ denotes the element obtained from w by applying the deleting operation $d-1$ times.

By [NT, Proposition 3.1], we have the following description of the non-abelian representation variety of $\mathfrak{b}(6n+1,3)$.

Lemma 3.2. *For the 2-bridge knot $\mathfrak{b}(6n+1,3)$, we have $w=(ab)^n(a^{-1}b^{-1})^n(ab)^n$ and*
$$R_w(x,z)=S_{3n}(z)-S_{3n-1}(z)-x^2(z-2)S^2_{n-1}(z)(S_n(z)-S_{n-1}(z)).$$

Let $r=waw^{-1}b^{-1}$. Then $\frac{\partial r}{\partial a}=w\left(1+(1-a)w^{-1}\frac{\partial w}{\partial a}\right)$ where
$$\frac{\partial w}{\partial a}=\left(1+(ab)^n(a^{-1}b^{-1})^n\right)\left(1+\cdots+(ab)^{n-1}\right)-(ab)^n\left(1+\cdots+(a^{-1}b^{-1})^{n-1}\right)a^{-1}.$$
Suppose $\rho:G_K\to SL_2(\mathbb{C})$ is a non-abelian representation. Then the twisted Alexander polynomial of K associated to ρ is
$$\Delta_{K,\rho}(t)=\det\Phi\left(\frac{\partial r}{\partial a}\right)\Big/\det\Phi(1-b)=\det\Phi\left(\frac{\partial r}{\partial a}\right)\Big/(1-tx+t^2).$$
We have
$$\begin{aligned}\det\Phi\left(\frac{\partial w}{\partial a}\right) &= t^{4n}\mid I+(I-tA)t^{-2n}(AB)^{-n}(BA)^n(AB)^{-n}\\ &\quad\times\{\left(I+(AB)^n(A^{-1}B^{-1})^n\right)\left(I+\cdots+(t^2AB)^{n-1}\right)\\ &\quad-(t^2AB)^n\left(I+\cdots+(t^{-2}A^{-1}B^{-1})^{n-1}\right)t^{-1}A^{-1}\}\mid\end{aligned}$$
where $A=\rho(a)$ and $B=\rho(b)$. Here $|M|$ denotes the determinant of M.

The following lemma follows easily.

Lemma 3.3. *The highest and lowest degree terms of $\Delta_{K,\rho}(t)$ are respectively $\mid I+A(AB)^{-n}(BA)^nA^{-1}\mid t^{4n}$ and $\mid I+(AB)^n(BA)^{-n}\mid t^0$.*

Let $\{S_j(z)\}_j$ be the sequence of Chebyshev polynomials defined by $S_0(z)=1$, $S_1(z)=z$, and $S_{j+1}(z)=zS_j(z)-S_{j-1}(z)$ for all integers j.

The following lemmas are standard, see e.g. [MT].

Lemma 3.4. *One has $S^2_j(z)-zS_j(z)S_{j-1}(z)+S^2_{j-1}(z)=1$.*

Lemma 3.5. *Suppose the sequence $\{M_j\}_j$ of 2×2 matrices satisfies the recurrence relation $M_{j+1}=zM_j-M_{j-1}$ for all integers j. Then*
$$M_j=S_{i-1}(z)M_1-S_{i-2}(z)M_0.$$

Lemma 3.6. *One has*
$$\mid I+(AB)^n(BA)^{-n}\mid=4+(z-2)(z+2-x^2)S^2_{n-1}(z).$$

Proof. By applying Lemma 3.5 twice, we have
$$\begin{aligned}\operatorname{tr}(AB)^n(BA)^{-n} &= \operatorname{tr}(AB)^n(A^{-1}B^{-1})^n\\ &= S^2_{n-1}(z)\operatorname{tr}ABA^{-1}B^{-1}+S^2_{n-1}(z)\operatorname{tr}I\\ &\quad-S_{n-1}(z)S_{n-2}(z)(\operatorname{tr}AB+\operatorname{tr}A^{-1}B^{-1}).\end{aligned}$$

From the identity $\operatorname{tr} CD = \operatorname{tr} C \operatorname{tr} D - \operatorname{tr} CD^{-1}$ for all matrices C, D in $SL_2(\mathbb{C})$, it is easy to see that $\operatorname{tr} ABA^{-1}B^{-1} = z^2 - zx^2 + 2x^2 - 2 = 2 + (z-2)(z+2-x^2)$. Hence, by Lemma 3.4, we obtain

$$\begin{aligned}\operatorname{tr}(AB)^n(BA)^{-n} &= (2+(z-2)(z+2-x^2))S_{n-1}^2(z) + 2S_{n-1}^2(z) - 2zS_{n-1}(z)S_{n-2}(z) \\ &= 2+(z-2)(z+2-x^2)S_{n-1}^2(z).\end{aligned}$$

The lemma follows since $\det(I+C) = 1 + \det C + \operatorname{tr} C$. □

3.0.1. *Genus.* Since the genus of $\mathfrak{b}(6n+1,3)$ is n, Lemmas 3.2, 3.3 and 3.6 imply that the number of conjugacy classes of non-abelian representations whose associated twisted Alexander polynomials do not determine the genus of $\mathfrak{b}(6n+1,3)$ is equal to the number of solutions $(x,z) \in \mathbb{C}^2$ of the following system

$$4 + (z-2)(z+2-x^2)S_{n-1}^2(z) = 0, \tag{3.1}$$

$$S_{3n}(z) - S_{3n-1}(z) - x^2(z-2)S_{n-1}^2(z)(S_n(z) - S_{n-1}(z)) = 0. \tag{3.2}$$

Suppose eq. (3.1) holds. Then $x^2(z-2)S_{n-1}^2(z) = 4 + (z^2-4)S_{n-1}^2(z)$. Eq. (3.2) is then equivalent to the following

$$S_{3n}(z) - S_{3n-1}(z) - (4+(z^2-4)S_{n-1}^2(z))(S_n(z) - S_{n-1}(z)) = 0. \tag{3.3}$$

Claim 3.7. *The equation* (3.3) *has n distinct roots, all of which are different from* ± 2.

Proof. Let $h_1(z)$ denote the left hand side of eq. (3.3). Since $S_j(2) = j+1$ and $S_j(-2) = (-1)^j(j+1)$ for all integers j, we have $h_1(2) = -3$ and $h_1(-2)S = (-1)^{n+1}(2n+3)$.

Suppose $z \neq \pm 2$. We may write $z = \alpha + \alpha^{-1}$ where $\alpha \neq \pm 1$. Since $S_j(z) = \frac{\alpha^{j+1} - \alpha^{-(j+1)}}{\alpha - \alpha^{-1}}$ for all integers j, we have $h_1(z) = \alpha^{-n}\frac{(2\alpha+1)\alpha^{2n}+\alpha+2}{\alpha+1}$. It follows that $h_1(z)$ is a polynomial in z of degree n. We want to show that $h_1(z)$ does not multiple roots. This holds true if we can show that the polynomial $h_2(\alpha) = 2\alpha^{2n+1} + \alpha^{2n} + \alpha + 2$ does not have multiple roots α such that $|\alpha| \geq 1$. (Note that if α is a root of h_2, then so is α^{-1}).

Suppose h_2 has a multiple root $\alpha \in \mathbb{C}$ such that $|\alpha| \geq 1$. Then $h_2(\alpha) = h_2'(\alpha) = 0$, or equivalently

$$(2\alpha+1)\alpha^{2n} + \alpha + 2 = 2((2n+1)\alpha + n)\alpha^{2n-1} + 1 = 0.$$

It follows that $\frac{\alpha+2}{2\alpha+1} = \frac{\alpha}{2((2n+1)\alpha+n)}$, i.e. $\alpha^2 + (\frac{5}{2} + \frac{3}{4n})\alpha + 1 = 0$. Hence $\alpha + \alpha^{-1} = -(\frac{5}{2} + \frac{3}{4n}) < -\frac{5}{2}$. Then $\alpha \in \mathbb{R}$, and since $|\alpha| \geq 1$ we must have $\alpha < -2$. Hence $h_1(\alpha) = (2\alpha+1)\alpha^{2n} + \alpha + 2 < 0$, a contradiction. The claim follows. □

For each solution z of eq. (3.3), we write $z = \alpha + \alpha^{-1}$ for some $\alpha \in \mathbb{C}^*$. Then from the proof of Claim (3.7), we have $\alpha \neq \pm 1$ and $\alpha^{-n}\frac{(2\alpha+1)\alpha^{2n}+\alpha+2}{\alpha+1} = 0$. It follows that $\alpha^{2n} = -\frac{\alpha+2}{2\alpha+1} \neq 1$. Eq. (3.1) is then equivalent to the following

$$x^2 = \frac{4+(z^2-4)S_{n-1}^2(z)}{(z-2)S_{n-1}^2(z)} = \frac{(\alpha+1)^2(\alpha^{2n}+1)^2}{\alpha(\alpha^{2n}-1)^2} = \frac{(\alpha-1)^2}{9\alpha},$$

i.e. $9x^2 = z - 2$. Hence Claim 3.7 implies that the system (3.1)+(3.2) has exactly $2n$ complex solutions (x,z). This means that the number of conjugacy classes of non-abelian representations whose associated twisted Alexander polynomials do not determine the genus of $\mathfrak{b}(6n+1,3)$ is equal to $2n$.

3.0.2. *Fiberedness.* The number of conjugacy classes of non-abelian representations whose associated twisted Alexander polynomials do not determine the fiberedness of $\mathfrak{b}(6n+1,3)$ is equal to the number of solutions $(x,z) \in \mathbb{C}^2$ of the following system of equations

$$3+(z-2)(z+2-x^2)S_{n-1}^2(z) = 0, \tag{3.4}$$

$$S_{3n}(z)-S_{3n-1}(z)-x^2(z-2)S_{n-1}^2(z)(S_n(z)-S_{n-1}(z)) = 0. \tag{3.5}$$

Suppose eq. (3.4) holds. Then $x^2(z-2)S_{n-1}^2(z) = 3+(z^2-4)S_{n-1}^2(z)$, and eq. (3.5) is equivalent to the following

$$S_{3n}(z)-S_{3n-1}(z)-(3+(z^2-4)S_{n-1}^2(z))(S_n(z)-S_{n-1}(z)) = 0. \tag{3.6}$$

Claim 3.8. *The equation* (3.6) *has n distinct roots, all of which are different from ± 2.*

Proof. Let $h_3(z)$ denote the left hand side of eq. (3.6). It is easy to see that $h_3(2)=-2$ and $h_3(-2)=2(-1)^{n+1}$. Suppose $z \neq \pm 2$. We may write $z=\alpha+\alpha^{-1}$ where $\alpha \neq \pm 1$. Then $h_3(z)=\alpha^{-n}(\alpha^{2n}+1)$, and the claim follows easily. □

For each solution z of eq. (3.6), we write $z=\alpha+\alpha^{-1}$ for some $\alpha \in \mathbb{C}^*$. Then from the proof of Claim (3.8), we have $\alpha \neq \pm 1$ and $\alpha^{2n}=-1$. Eq. (3.4) is then equivalent to the following

$$x^2 = \frac{3+(z^2-4)S_{n-1}^2(z)}{(z-2)S_{n-1}^2(z)} = \frac{(\alpha+1)^2(\alpha^{4n}+\alpha^{2n}+1)}{\alpha(\alpha^{2n}-1)^2} = \frac{(\alpha+1)^2}{4\alpha},$$

i.e. $4x^2=z+2$. Hence Claim 3.8 implies that the system (3.4)+(3.5) has exactly $2n$ complex solutions (x,z). This means that the number of conjugacy classes of non-abelian representations whose associated twisted Alexander polynomials do not determine the fiberedness of $\mathfrak{b}(6n+1,3)$ is equal to $2n$.

This completes the proof of Theorem 1.

Remark 3.9. One can easily show that all the solutions (x,z) of the systems (3.1)+(3.2) and (3.4)+(3.5) satisfying the condition $x \neq 2$. It follows that the twisted Alexander polynomial associated to any parabolic $SL_2(\mathbb{C})$-representation of the knot group of $\mathfrak{b}(6n+1,3)$ determines the fiberedness and genus of $\mathfrak{b}(6n+1,3)$. This gives another proof of [MT, Theorem 1.1] and hence of a conjecture of Dunfield, Friedl and Jackson [DFJ] for $\mathfrak{b}(6n+1,3)$. (Note that $\mathfrak{b}(6n+1,3)=J(3,2n)$ in the notation of [MT].)

4. Proof of Theorem 2

We first note that the m-twist knot K_m, $m>0$, is non-fibered if and only if $m>2$. (K_1 is the trefoil knot, and K_2 is the figure-8 knot.)

4.1. K_{2n}, $n>1$. From [Tr] (and also [NT]), the knot group of $K=K_{2n}$ is $\langle a,b \mid wa=bw\rangle$ where a,b are meridians depicted in Figure 1 and $w=(ba^{-1})^n b(ab^{-1})^n$. Note that this is not the standard presentation of a two-bridge knot.

For a representation $\rho: G_K \to SL_2(\mathbb{C})$, let $x=\operatorname{tr}\rho(a)=\operatorname{tr}\rho(b)$ and $y=\operatorname{tr}\rho(ab^{-1})$. From [Tr], we have the following

Lemma 4.1. *A representation $\rho: G_K \to SL_2(\mathbb{C})$ is non-abelian if and only if $(x,y)\in\mathbb{C}^2$ is a root of the polynomial*

$$R_{even}(x,y) = (y+1)S_{n-1}^2(y)-S_n^2(y)-2S_{n-1}(y)S_n(y)+x^2S_{n-1}(y)(S_n(y)-S_{n-1}(y)).$$

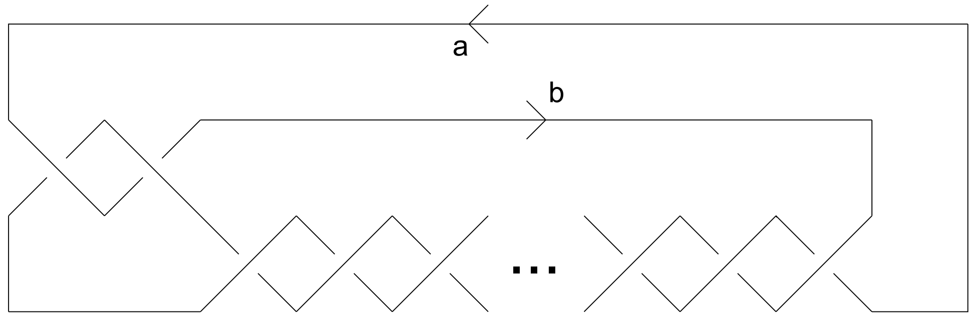

FIGURE 1. K_{2n}, $n > 0$.

Let $r = waw^{-1}b^{-1}$. Then $\frac{\partial r}{\partial a} = w\left(1 + (1-a)w^{-1}\frac{\partial w}{\partial a}\right)$ where

$$\frac{\partial w}{\partial a} = -(1 + \cdots + (ba^{-1})^{n-1})ba^{-1} + (ba^{-1})^n b(1 + \cdots + (ab^{-1})^{n-1}).$$

Suppose $\rho : G_K \to SL_2(\mathbb{C})$ is a non-abelian representation. Then the twisted Alexander polynomial of K associated to ρ is

$$\Delta_{K,\rho}(t) = \det \Phi\left(\frac{\partial r}{\partial a}\right) / \det \Phi(1-b) = \det \Phi\left(\frac{\partial r}{\partial a}\right) / (1 - tx + t^2).$$

We have

$$\begin{aligned} \det \Phi\left(\frac{\partial w}{\partial a}\right) &= t^2 |I + (I - tA)t^{-1}(BA^{-1})^n B^{-1}(AB^{-1})^n \\ &\quad \times \{-(1 + \cdots + (BA^{-1})^{n-1})BA^{-1} + (BA^{-1})^n tB(1 + \cdots + (AB^{-1})^{n-1})\}|. \end{aligned}$$

where $A = \rho(a)$ and $B = \rho(b)$.

Let $\{T_j(z)\}_j$ be the sequence of Chebyshev polynomials defined by $T_0(z) = 2$, $T_1(z) = z$, and $T_{j+1}(z) = zT_j(z) - T_{j-1}(z)$ for all integers j.

The following lemma follows easily.

Lemma 4.2. *The highest and lowest degree terms of $\Delta_{K,\rho}(t)$ are respectively $|1 + \cdots + (AB^{-1})^{n-1}|t^4 = \frac{T_n(y)-2}{y-2}\, t^4$ and $|-(1 + \cdots + (BA^{-1})^{n-1})|t^0 = \frac{T_n(y)-2}{y-2}\, t^0$.*

4.1.1. *Genus.* Since the genus of K_{2n} is 1, Lemmas 4.1 and 4.2 imply that the number of conjugacy classes of non-abelian representations whose associated twisted Alexander polynomials do not determine the genus of K_{2n} is equal to the number of solutions $(x, y) \in \mathbb{C}^2$ of the following system

$$(T_n(y) - 2)/(y - 2) = 0, \tag{4.1}$$

$$(y+1)S_{n-1}^2(y) - S_n^2(y) - 2S_{n-1}(y)S_n(y) + x^2 S_{n-1}(y)(S_n(y) - S_{n-1}(y)) = 0. \tag{4.2}$$

Suppose eq. (4.1) holds. $\frac{T_n(y)-2}{y-2} = 0$. Then $y \neq 2$ and $T_n(y) = 2$. We write $y = \beta + \beta^{-1}$ where $\beta \neq 1$. Then $T_n(y) = \beta^n + \beta^{-n} = 2$ is equivalent to $\beta^n = 1$, or equivalent $\beta = e^{i\frac{2\pi k}{n}}$ for some $1 \leq k \leq \frac{n}{2}$. It follows that eq. (4.1) has $\lfloor n/2 \rfloor$ distinct solutions given by $y = 2\cos\frac{2k\pi}{n}$ where $1 \leq k \leq \frac{n}{2}$.

If $y = 2\cos\frac{2k\pi}{n}$ for some $1 \leq k < \frac{n}{2}$ then $y = \beta + \beta^{-1}$ where $\beta = e^{i\frac{2k\pi}{n}}$. It follows that $S_{n-1}(y) = 0$ and $S_n(y) = 1$. Hence R_{even}, the left hand side of eq. (4.2), is equal to -1.

If $y=-2$ (in this case n must be even), it is easy to see that $R_{even}=-(2n+1)(nx^2+2n+1)$, since $S_j(-2)=(-1)^j(j+1)$ for all integers j. Then eq. (4.2) is equivalent to $x^2=-(2+\frac{1}{n})$.

Hence the system (4.1)+(4.2) has exactly $1+(-1)^n$ solutions.

4.1.2. *Fiberedness.* The number of conjugacy classes of non-abelian representations whose associated twisted Alexander polynomials do not determine the fiberedness of K_{2n} is equal to the number of solutions $(x,y)\in\mathbb{C}^2$ of the following system

$$(T_n(y)-2)/(y-2) = 1, \tag{4.3}$$

$$(y+1)S^2_{n-1}(y)-S^2_n(y)-2S_{n-1}(y)S_n(y)+x^2S_{n-1}(y)(S_n(y)-S_{n-1}(y)) = 0. \tag{4.4}$$

Suppose eq. (4.3) holds. Then $y\neq 2$ and $T_n(y)=y$. We write $y=\beta+\beta^{-1}$ where $\beta\neq 1$. Then $T_n(y)=y$ is equivalent to $\beta^{n+1}=1$ or $\beta^{n-1}=1$. It follows that the distinct solutions of eq. (4.3) are $y=2\cos\frac{2k\pi}{n+1}$ where $1\le k<\frac{n+1}{2}$, $y=2\cos\frac{2k\pi}{n-1}$ where $1\le k<\frac{n-1}{2}$, and (if n is odd) $y=-2$.

If $y=-2$ (in this case n must be odd), it is easy to see that eq. (4.4) is equivalent to $x^2=-(2+\frac{1}{n})$.

Suppose $y=2\cos\frac{2k\pi}{n+1}$ where $1\le k<\frac{n+1}{2}$. Then $y=\beta+\beta^{-1}$ where $\beta=e^{i\frac{2k\pi}{n+1}}$. It follows that $S_{n-1}(y)=-1$ and $S_n(y)=0$. Hence $R_{even}=-(x^2+1)$, and eq. (4.4) is equivalent to $x^2=-1$.

Suppose $y=2\cos\frac{2k\pi}{n-1}$ where $1\le k<\frac{n-1}{2}$. Then $y=\beta+\beta^{-1}$ where $\beta=e^{i\frac{2k\pi}{n-1}}$. It follows that $S_{n-1}(y)=1$ and $S_n(y)=y$. Hence $R_{even}=-(y^2+y-1)+x^2(y-1)$. If $y=1$, i.e. $k=\frac{n-1}{6}$, then $R_{even}=-1$. If $y\neq 1$ then eq. (4.4) is equivalent to $x^2=\frac{y^2+y-1}{y-1}$. If $y=-\frac{1+\sqrt{5}}{4}$ (i.e. $k=\frac{2(n-1)}{5}$) or $y=\frac{-1+\sqrt{5}}{4}$ (i.e. $k=\frac{n-1}{5}$) then $y^2+y-1=0$, and eq. (4.4) is equivalent to $x=0$.

Hence the system (4.3)+(4.4) has exactly $2n-2-a_n-b_n$ solutions, where a_n,b_n are as defined in Theorem 2(i).

4.2. K_{2n-1}, $n>1$**.** From [Tr] (and also [NT]), the knot group of $K=K_{2n-1}$ is $\langle a,b \mid wa=bw\rangle$ where a,b are meridians depicted in Figure 2 and $w=(ab^{-1})^nb(ba^{-1})^n$. Note that this is not the standard presentation of a two-bridge knot.

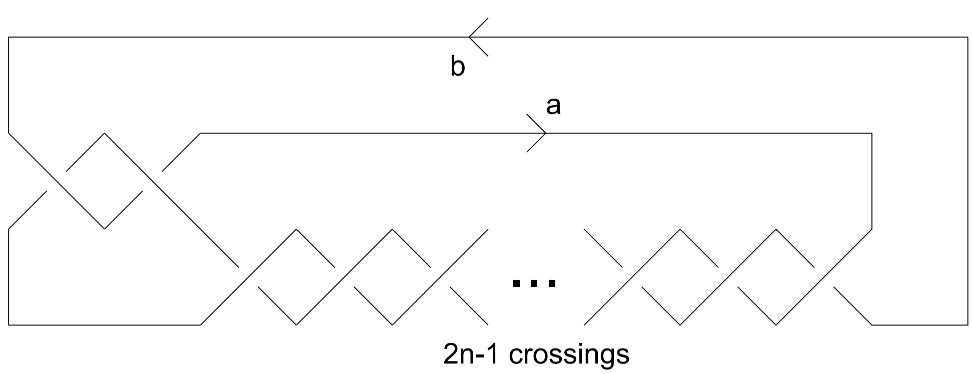

FIGURE 2. K_{2n-1}, $n>0$.

For a representation $\rho: G_K\to SL_2(\mathbb{C})$, let $x=\operatorname{tr}\rho(a)=\operatorname{tr}\rho(b)$ and $y=\operatorname{tr}\rho(ab^{-1})$. From [Tr], we have the following

Lemma 4.3. *A representation $\rho : G_K \to SL_2(\mathbb{C})$ is non-abelian if and only if $(x, y) \in \mathbb{C}^2$ is a root of the polynomial*

$$R_{odd}(x,y) = -(y+1)S_{n-1}^2(y)+S_{n-2}^2(y)+2S_{n-1}(y)S_{n-2}(y)+x^2S_{n-1}(y)(S_{n-1}(y)-S_{n-2}(y)).$$

Let $r = waw^{-1}b^{-1}$. Then $\frac{\partial r}{\partial a} = w\left(1 + (1-a)w^{-1}\frac{\partial w}{\partial a}\right)$ where

$$\frac{\partial w}{\partial a} = (1 + \cdots + (ab^{-1})^{n-1}) - (ab^{-1})^n b(1 + \cdots + (ba^{-1})^{n-1})ba^{-1}.$$

Suppose $\rho : G_K \to SL_2(\mathbb{C})$ is a non-abelian representation. Then the twisted Alexander polynomial of K associated to ρ is

$$\Delta_{K,\rho}(t) = \det \Phi\left(\frac{\partial r}{\partial a}\right) \Big/ \det \Phi(1-b) = \det \Phi\left(\frac{\partial r}{\partial a}\right) \Big/ (1 - tx + t^2).$$

We have

$$\begin{aligned} \det \Phi\left(\frac{\partial w}{\partial a}\right) &= t^2|I + (I - tA)t^{-1}(AB^{-1})^n B^{-1}(BA^{-1})^n \\ &\quad \times\{(1 + \cdots + (AB^{-1})^{n-1}) - (AB^{-1})^n tB(1 + \cdots + (BA^{-1})^{n-1})BA^{-1}\}|. \end{aligned}$$

where $A = \rho(a)$ and $B = \rho(b)$. The following lemma follows easily.

Lemma 4.4. *The highest and lowest degree terms of $\Delta_{K,\rho}(t)$ are respectively $|1 + \cdots + (BA^{-1})^{n-1}|t^4 = \frac{T_n(y)-2}{y-2}\, t^4$ and $|-(1 + \cdots + (AB^{-1})^{n-1})|t^0 = \frac{T_n(y)-2}{y-2}\, t^0$.*

Applying Lemmas 4.3 and 4.4, the proof of Theorem 2(ii) is similar to that of Theorem 2(i). This completes the proof of Theorem 2.

Remark 4.5. From the proof of Theorem 2, one can easily see that the twisted Alexander polynomial associated to any parabolic $SL_2(\mathbb{C})$-representation of the knot group of the m-twist knot K_m, $m > 0$, determines the fiberedness and genus of K_m. This gives another proof of [MT, Theorem 1.1] and hence of a conjecture of Dunfield, Friedl and Jackson [DFJ] for K_m.

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